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Unsteady transonic flows in two-dimensional channels

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A two-dimensional, unsteady, transonic, irrotational, inviscid flow of a perfect gas with constant specific heats is considered. The analysis involves perturbations from a uniform sonic isentropic flow. The governing perturbation potential equations are derived for various orders of the ratio of the characteristic time associated with a temporal flow disturbance to the time taken by a sonic disturbance to traverse the transonic regime. The case where this ratio is large compared to one is studied in detail. A similarity solution involving an arbitrary function of time is found and it is shown that this solution corresponds to unsteady channel flows with either stationary or time-varying wall shapes. Numerical computations are presented showing the temporal changes in flow structure as a disturbance dies out exponentially for the following typical nozzle flows: simple accelerating (Meyer) flow and flow with supersonic pockets (Taylor and limiting Taylor flow).

1. Introduction

Steady transonic flow in nozzles has received a great deal of attention, and extensive descriptions of both inviscid and viscous flows are contained in the books by Guderley (1962) and Ferrari & Tricomi (1968) and the review paper by Sichel (1968), for example. Until recently, unsteady transonic flow studies had been confined essentially to wing and body flows where conditions are such that, generally, the linearized equations are applicable (Landahl 1961, 1962). On the other hand, studies of the unsteady potential equation for transonic flow by Lin, Reissner & Tsien (1948) and later by Timman (1962) involve the same principles in ordering the various terms as those employed here. Recently, time-dependent computer solutions have been obtained for some cases of transonic flow in turbomachine cascades (e.g. Oliver & Sparis 1971).

Transonic nozzle flow solutions have applications in turbo-machinery as well as in de Laval nozzle and channel flows. For example, for certain parameter ranges one can consider two streamlines on the same side of the axis as walls, so that the solution represents flow in a curved channel corresponding to the region between two adjacent blades in a cascade. It seems apparent, then, that, just as in the steady-state case, unsteady nozzle solutions will be of considerable fundamental and practical interest. As in unsteady wing theory there are several characteristic time regimes, each associated with a different class of physical problems. In the following, the governing perturbation potential equation is derived for each regime, and solutions are given for one of these. The flow is assumed to be inviscid, two-dimensional, compressible and transonic; the gas is assumed to follow the perfect gas law and to have constant specific heats.

2. Derivation of equations

The problem considered is that of perturbations from a steady, sonic, irrotational, two-dimensional stream flowing in the X direction. Thus, a velocity potential $\Phi(X, Y, T)$ exists, Φ being made dimensionless with respect to a characteristic length $\bar{L}$ and the sonic speed $\bar{a}^*$; the bar indicates a dimensional quantity. X and Y are co-ordinate distances made dimensionless with respect to $\bar{L}$, and T is the time, made dimensionless with respect to $\bar{L}/\bar{a}^*$.

The independent variables are stretched as follows, with the orders of the gauge parameters to be given later.

$$X = \delta x, \quad \delta = \bar{L}_x/\bar{L}, \qquad x = O(1), \tag{1a}$$

$$Y = \epsilon y, \quad \epsilon = \bar{L}_y/\bar{L}, \qquad y = O(1), \tag{1b}$$

$$T = \tau t, \quad \tau = \bar{T}_{ch}\bigg/\left(\frac{\bar{L}}{\bar{a}^*}\right), \quad t = O(1). \tag{1c}$$

In (1), $\bar{L}_x$ and $\bar{L}_y$ are fictitious lengths of the order of the physical extent of the transonic region in the X and Y directions respectively. Hence, in the transonic region, x and y, which are co-ordinate distances made dimensionless with respect to $\bar{L}_x$ and $\bar{L}_y$ respectively, are of order unity. Likewise, t, which is the time made dimensionless with respect to $\bar{T}_{ch}$, the characteristic time associated with the disturbance, is of order unity.

Finally, a first-order perturbation potential function ϕ is defined such that

$$\Phi(X, Y, T) = X + E_1\delta\phi(x, y, t) + \ldots, \tag{2}$$

where $E_1 \ll 1$, $\phi_x = O(1)$ and δ is at most $O(1)$. That is,

$$\Phi_X = U = 1 + E_1\phi_x + \ldots, \tag{3}$$

where U is the velocity component in the X direction. Hence, $E_1 = O(U-1)$ is a measure of the deviation of the flow from the sonic velocity.

If the stretched co-ordinates defined in (1) and the perturbation potential defined in (2) are substituted into the so-called general gas dynamic equation and Bernoulli's equation (Guderley 1962), a governing equation for ϕ can be derived (Adamson 1971):

$$-E_1(\gamma+1)\phi_x\phi_{xx} + \frac{\delta^2}{\epsilon^2}\phi_{yy} - 2\frac{\delta}{\tau}\phi_{xt} - \frac{\delta^2}{\tau^2}\phi_{tt} = 0. \tag{4}$$

The condition that δ/ϵ is at most of order one has been used in deriving this equation. This condition is certainly met in nozzle flows; in fact, since it is desired that the equation should reduce to the usual two-dimensional equations for steady flow ($\tau \to \infty$), it is seen that $\delta^2/\epsilon^2 = O(E_1)$ must hold. In addition, the characteristic length $\bar{L}$ is chosen here to coincide with the y extent of the transonic region, corresponding to the order of the throat half width, so that $\epsilon = O(1)$.

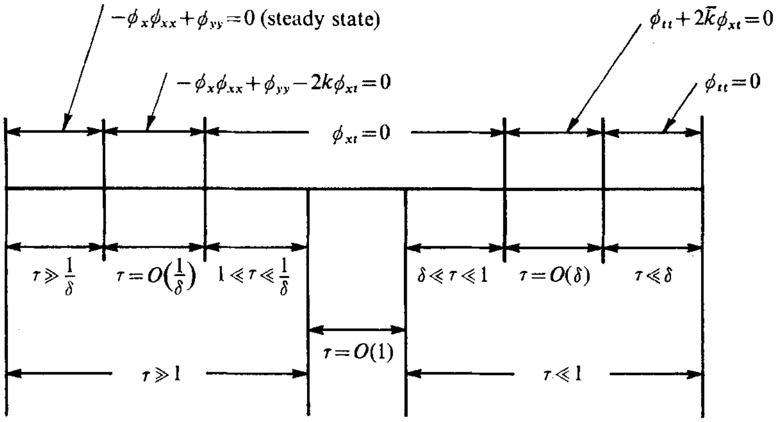

FIGURE 1. Perturbation potential equations for each given order of τ relative to δ. k and $\bar{k}$ are arbitrary constants of $O(1)$.

For convenience, then, one can set

$$\epsilon = 1, \quad \delta^2 = (\gamma+1)E_1. \qquad (5a),(5b)$$

Equation (4) then becomes

$$-\phi_x\phi_{xx}+\phi_y-\frac{1}{\tau^2}\phi_{tt}-\frac{2}{\tau\delta}\phi_{xt} = 0. \qquad (6)$$

It can be seen from (6) that there are several different parameter ranges, or distinguished limits, each with a different governing equation. The various τ ranges relative to δ, and the corresponding governing equations, are illustrated in figure 1 and hold as long as the X scale (δ) and time scale (τ) can be specified separately. The physical situations corresponding to each of the distinguished limits are as follows.

(i) $\tau = O(\delta)$: $\bar{T}_{ch}$ is of the order of the time it takes a sonic disturbance to cross the transonic region in the flow direction. $\bar{T}_{ch} = O(\bar{L}_x/\bar{a}^*)$.

(ii) $\tau = O(1)$: $\bar{T}_{ch}$ is of the order of the time it takes a sonic disturbance to cross the transonic region in the transverse direction. $\bar{T}_{ch} = O(\bar{L}/\bar{a}^*)$.

(iii) $\tau = O(1/\delta)$: $\bar{T}_{ch}$ is large compared with the time it takes a sonic disturbance to cross the transonic region in either direction.

$$\bar{T}_{ch} = O\left(\frac{\bar{L}}{\bar{a}^*}\frac{\bar{L}}{\bar{L}_x}\right), \quad \text{so} \quad \bar{T}_{ch} \gg \frac{\bar{L}}{\bar{a}^*} \gg \frac{\bar{L}_x}{\bar{a}^*}.$$

Each of the above τ regimes corresponds to a different class of physical problems. For example, it can be shown (Adamson 1971) that many practical problems associated with unsteady flows through aircraft jet engines and thrustor nozzles fall in the regime $\tau = O(1/\delta)$. Moreover, this τ regime is that nearest to steady-state conditions and it should, therefore, be possible to study solutions which tend to steady-state solutions as $t\to\infty$. For these reasons, it was decided to study this regime; solutions are discussed in following sections.

3. Definition of channel wall

The channel wall is treated here in the same fashion as in unsteady wing theory. That is, along the wall, which may in general be moving in time,

$$Y_w = F_w(X, T). \tag{7}$$

Since the general flow is a perturbation of a uniform sonic flow (7) can be written in stretched co-ordinates as

$$y_w = y_i + w f_w(x, t), \tag{8}$$

where y_i is a constant and $w \ll 1$. Now, along the wall, $y = y_w$ and the Eulerian derivative of $y - y_w$ is certainly also zero there:

$$\frac{\mathscr{D}}{\mathscr{D}T}(y - y_w) = 0. \tag{9}$$

After stretching the variables and using for the velocity components

$$U = \Phi_X, \quad u = \phi_x, \tag{10a}$$

$$V = \Phi_Y, \quad v = \phi_y, \tag{10b}$$

(9) becomes,

$$-\frac{w}{\tau}\frac{\partial f_w}{\partial t} - \frac{w}{\delta}\frac{\partial f_w}{\partial x} + \frac{\delta^3}{(\gamma+1)} v_w = 0. \tag{11}$$

Thus the equations for the wall for the three τ regimes mentioned previously may be obtained from (11). For example, for $\tau = O(1/\delta)$, say

$$\tau = 1/k\delta, \tag{12}$$

where k is an arbitrary constant of order one, it is seen that the second term of (11) is large compared with the first and that therefore $w = O(\delta^4)$. For convenience, w is defined so as to simplify (11). Thus, the expression for w, equation (11) and equation (8) become respectively

$$w = \delta^4/(\gamma+1), \tag{13a}$$

$$\partial f_w/\partial x = v_w, \tag{13b}$$

$$y_w = y_i + (\gamma+1) E_1^2 f_w(x, t). \tag{13c}$$

Therefore, for $\tau = O(1/\delta)$ the wall is a streamline at any given instant and variations of the streamline shape from a constant value of y are of second order.

4. Critical velocity

In steady-state flow problems Bernoulli's equation reduces to the simple relation that the stagnation enthalpy is everywhere constant. Then it is easily shown that the critical velocity is constant. This is not necessarily the case in unsteady flow problems. For example, in this unsteady nozzle flow problem the critical velocity may or may not be constant, depending on the τ regime considered. Thus, by definition, $U^2 + V^2 = a^2$ at the critical condition, where a is the dimensionless sound speed, and using equations (2) and (10) in Bernoulli's equation, one can show that to first order

$$2\phi_x = -(\gamma - 1)(\phi_x + (\delta/\tau)\phi_t). \tag{14}$$

Hence, for $\tau = O(1/\delta)$ or $\tau = O(1)$ the term involving ϕ_t is of higher order, since $\delta \ll 1$, and the resulting condition is $\phi_x = 0$ (i.e. $u = 0$) to first order. Then $U = 1$, or $\bar{U} = \bar{a}^*$ and the critical velocity is constant and equal to the parent flow velocity. On the other hand, if $\tau = O(\delta)$, $u = \phi_x$ depends on ϕ_t, so the critical velocity varies with time even at first order.

In view of the above remarks, it is clear that for $\tau = O(1/\delta)$ the sonic line is defined by $u = 0$.

5. Solutions for $\tau = O(1/\delta)$

The perturbation potential equation for the regime $\tau = O(1/\delta)$ is shown in figure 1, where τ is actually related to δ as in equation (12). This equation can be written in terms of the perturbation velocity components defined in (10). The resulting equation may be combined with the equation expressing the condition of irrotationality to give an equation in terms of u alone. Thus

$$u_x^2 + uu_{xx} - u_{yy} + 2ku_{xt} = 0. \tag{15}$$

The equation is written in this form to facilitate comparison with steady-state methods of solution.

At this point a similarity solution is sought for (15). This is done because of the success of such methods in studying steady inviscid and viscous transonic nozzles flows. Of course, with a similarity solution, one does not obtain a general solution to the equations from which a specific solution may be derived through the application of boundary and/or initial conditions. Instead, one obtains a self-similar solution which satisfies only very special boundary conditions, which, for example, may or may not correspond to the physical problem in hand. In nozzle flow problems, similarity solutions which correspond to steady expansion through nozzles have been found both for inviscid flows (Tomotika & Tamada 1950) and for flows with longitudinal viscous effects (Sichel 1966). Hence it seemed possible that physically plausible unsteady similarity solutions exist, and this turned out to be the case. Thus, a similarity solution may be derived which is an extension of that found by Tomotika & Tamada (1950) and employed by Sichel (1966) in their studies of steady transonic nozzle flow.

Let
$$s = x + by^2 + \beta(t), \tag{16a}$$

$$u = Z(s) + 4b^2y^2 - 2k\beta', \tag{16b}$$

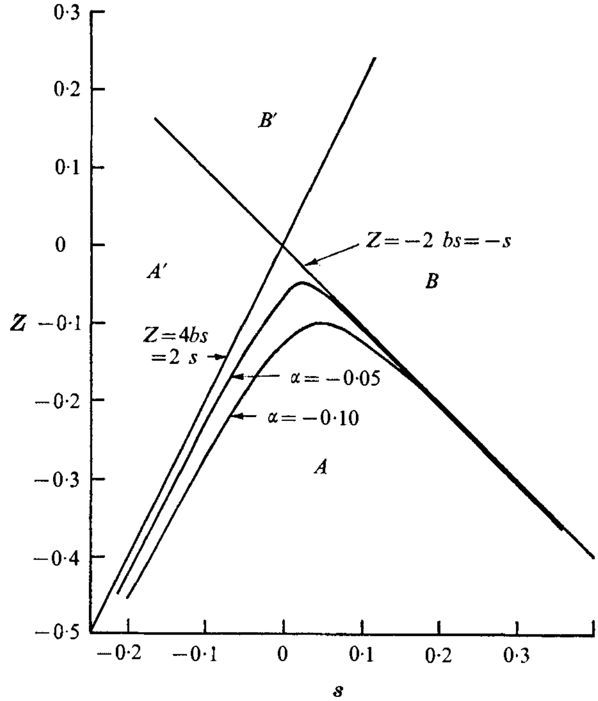

FIGURE 2. Z *vs.* s from the solution of Tomotika & Tamada (1950) for $b = \frac{1}{2}$ and various α.

where b is a constant, $\beta(t)$ is an arbitrary function of time and the prime on β indicates the derivative with respect to time. Then from the irrotationality condition, $u_y - v_x = 0$, it can be shown that

$$v = 2byZ + 8b^2yx + \tfrac{1}{3}8b^3y^3 + y(8b^2\beta - 4k^2\beta''). \tag{17}$$

Further, if equations (16) are substituted into (15), it is found that a *similarity* solution does indeed exist, i.e. that (15) reduces to the ordinary nonlinear differential equation for $Z(s)$:

$$ZZ'' + (Z' - 4b)(Z' + 2b) = 0, \tag{18}$$

the solution for which is

$$(Z - 4bs)^2(Z + 2bs) = \alpha^3/4b^3, \tag{19}$$

where α is a constant of integration, the physical meaning of which will become apparent later. Thus, we have the remarkable result that an analytical solution involving an arbitrary function of time exists for the unsteady transonic nozzle flow problem. The form of this function of time is related to the boundary conditions of the problem under consideration, as will be shown.

Equation (18) is precisely the equation studied by Tomotika & Tamada (1950) and so provides here the unsteady counterpart of their steady flow discussions. For example, the special solution $Z = 4bs$ ($\alpha = 0$) corresponds to Meyer flow, a simple expansion from subsonic to supersonic flows, and in the present case one is able to study various cases of unsteady expansive flows, depending on β. For $\alpha < 0$, the flow is the so-called Taylor flow, where pockets of supersonic flow exist at the walls, and the special solution $Z = 4bs$ for $s < 0$ and $Z = -2bs$ for $s > 0$ corresponds to limiting Taylor flow ($\alpha \to 0$ from $a < 0$) where the supersonic pockets meet at the centre-line. Again, the present solution allows consideration of the unsteady counterpart of these flows. Explicit solutions for z in terms of s may be found by solving the cubic equation for z given in (19) (Adamson 1971).

Typical curves of z *vs.* s are shown in figure 2 for $\alpha = -0{\cdot}05$ and $\alpha = -0{\cdot}10$. As shown by Tomotika & Tamada (1950) and Sichel (1966), similar curves exist in the regions indicated in figure 2 by the letters A', B' and B. However, regions A' and B are not physically meaningful and the flows represented by the solutions in region B' are entirely supersonic and thus not of interest here.

It can be shown that for the solution curves shown in region A the maximum point on a curve of α = constant occurs at

$$Z_m = \alpha/2b, \quad s_m = -\alpha/8b^2. \tag{20a, b}$$

Equations (20) and (16) can then be used to find the location (i.e. x and y values) of the minimum point of a curve of constant u at a given time. Therefore, this point is determined by the values of b, k, and α.

It is apparent from (13*b*), and the form of the similarity solution, that if it is desired to study a flow through a channel with rigid walls then v must be independent of time, i.e. all streamlines must be independent of time. This is possible only for very special solutions and a given form for β. On the other hand, the variations of the wall from a constant value of y are of second order in E_1, as is always true in transonic flow (e.g. equation (13*c*)), and wall fluctuations will therefore be small, if general functions of time are considered for β. Finally, it is probably true that the blades in compressors and turbines do twist and vibrate, and here one can consider the solutions as those associated with various wall motions.

5.1. *Unsteady Meyer flow*

A simple acceleration of the flow from subsonic to supersonic velocities is generally referred to as Meyer flow. As mentioned previously, this flow is given by the simple solution for $\alpha = 0$:

$$Z = 4bs. \tag{21}$$

The corresponding velocity components are found by substituting (21) and (16*a*) into (16*b*) and (17).

Solutions will be presented here for the case where the nozzle walls are rigid, and hence for the case where v is independent of time. It is readily shown that for this to be the case.

$$\beta = c_1 e^{2bt/k} + c_2 e^{-2bt/k} - c_3, \tag{22}$$

where c_1, c_2, and c_3 are arbitrary constants. With this form of β, u and v are

$$u = 4b(x-x_0) + 8b^2y^2 + 8bc_2 e^{-2bt/k}, \tag{23a}$$

$$v = y[16b^2(x-x_0) + \tfrac{32}{3}b^3y^2], \tag{23b}$$

where $x_0 = c_3$ is the value of the x co-ordinate along the axis at the sonic line ($u = 0$) in the steady-state limit as $t \to \infty$. It should be noted that (23) reduces to the proper steady-state solution in this limit. It should also be noted that the $\exp(2bt/k)$ term in (22) does not appear in u. This leads to an important result. First, $k > 0$ by definition; next, it is seen from (21) and (16) that $u_x = 4b$. Hence the sign of b determines whether the flow is accelerating ($b > 0$) or decelerating ($b < 0$). However, it is seen from (23) that for $b < 0$ and $k > 0$ the unsteady term increases without limit as t increases. Since an exponential with a positive argument does not appear, it is not possible, say, to set $c_2 = 0$ and retain the c_1 term in (22) and study decelerating flows. Thus, for the nozzle wall shape corresponding to this solution, simple decelerating unsteady flows are impossible; this may be an indication that a shock must form.

As mentioned previously, for the given τ regime, the wall is a streamline at any given instant, and, of course, any streamline can be considered as the channel wall. However, for this example, it is of interest to choose the wall streamline using the same conditions as those employed by Tomotika & Tamada (1950) and Sichel (1966), so that results can be compared. Hence, in this case, the streamline with a prescribed ratio of nozzle half width $\bar{h}$ to radius of curvature at the throat $[\bar{Y}_{\bar{X}\bar{X}}]_{th}^{-1}$ is chosen to be the wall. Then the throat co-ordinates can be determined by prescribing this ratio and using the condition that $v = 0$ at the throat. Thus, if $\mathscr{R}_{th}$ is designated as this dimensionless radius of curvature at the throat ($\mathscr{R}_{th} = [\bar{h}\bar{Y}_{\bar{X}\bar{X}}]_{th}^{-1}$) then $\mathscr{R}_{th}$ can be related to b, $\bar{h}$, and $\bar{L}$ through equations (13) and the throat conditions, and u and v can be written in terms of $\mathscr{R}_{th}$. Furthermore, since a streamline is defined at any instant, to first order, as

$$\frac{\partial Y_{SL}}{\partial X} = \frac{1}{\delta}\frac{\partial y_{SL}}{\partial x} = \frac{V}{U} = (\gamma+1)^{\frac{1}{2}} E_1^{\frac{3}{2}} v \tag{24}$$

it is easy to show that in this case the equation for a streamline is

$$\left(\frac{\bar{Y}}{\bar{h}}\right)_{SL} = \frac{\bar{Y}_{\min}}{\bar{h}}\left[1 + (2\mathscr{R}_{th})^{-1}\left(\frac{\bar{X}}{\bar{h}} - \frac{\bar{X}_{\min}}{\bar{h}}\right)\right], \tag{25}$$

where $\bar{X}_{\min}$ and $\bar{Y}_{\min}$ are the physical co-ordinates of the minimum point of any streamline. In deriving (25) from (24), it is sufficient to replace Y by $Y_{\min}$ in the equation for v since the equation for a streamline is simply that Y equals a constant plus a small quantity (see equation (13*c*)). A streamline shape is calculated by choosing a value of $\bar{Y}_{\min}/\bar{h}$, finding the corresponding $\bar{X}_{\min}/\bar{h}$ from the equation $v = 0$ and using these values in (25). The wall corresponds to $\bar{Y}_{\min} = \bar{h}$.

Typical calculations for the upper half plane are shown in figure 3 for $b = k = \frac{1}{2}$, $\mathscr{R}_{th} = 4$, $\gamma = 1{\cdot}4$, $E_1 = 0{\cdot}1$ and $(\bar{X}_0/\bar{h}) = -0{\cdot}1549$, values which are those used by Tomotika & Tamada (1950) and Sichel (1966). In addition $c_2 = \pm 0{\cdot}8068$ for this calculation. The wall and a typical streamline are shown. Also, the position of the sonic line ($u = 0$) at various times, for c_2 positive and negative, is illustrated. In each case the flow is subsonic upstream of the sonic line and supersonic downstream of the sonic line. It should be noted that although $\bar{X}_0/\bar{h}$ may be chosen arbitrarily the position of the sonic lines does not change relative to the throat as $\bar{X}_0/\bar{h}$ is varied.

It can readily be shown that the pressure perturbation is proportional to the

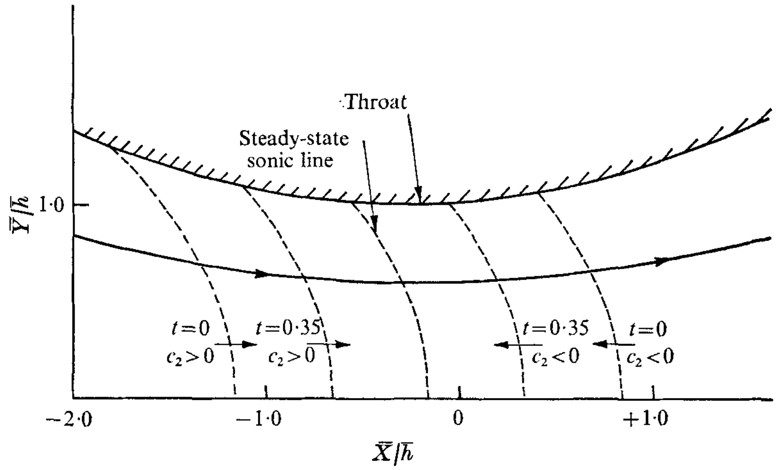

FIGURE 3. Solution for unsteady Meyer flow with stationary walls for $\tau = (k\delta)^{-1}$, $b = k = \frac{1}{2}$, $\gamma = 1{\cdot}4$, $E_1 = 0{\cdot}1$, $\mathscr{R}_{th} = 4$, β as given in equation (22), and $c_2 = \pm 0{\cdot}8068$. ----, sonic lines; —>—, streamline; arrows indicate motion of the sonic line as time increases. Flow is subsonic upstream and supersonic downstream of the sonic line in each case.

velocity perturbation but of opposite sign. Hence, it can be seen that the solutions for t finite and $c_2 > 0$ represent a flow returning to a steady-state condition as the upstream pressure recovers from a decrease in its steady-state value. The solutions for t finite and $c_2 < 0$ represent a flow returning to the steady state as the upstream pressure recovers from an increase over its steady-state value. The remarkable feature of the solutions is that the sonic line moves away from the throat in unsteady flow. Hence, for $c_2 > 0$ and $t = 0$, for example, the flow is supersonic in an area smaller than that corresponding to the sonic surface. As noted previously, the flow structure between the wall and the indicated streamline corresponds to the asymmetric channel flow found in the channel between two adjacent blades in a cascade.

It should be noted that in figure 3 some liberties have been taken with the solution to illustrate the important features; actually, the X extent of the first-order solution should not be as large as that shown.

5.2. *Limiting Taylor flow*

For the case referred to as limiting Taylor flow, $\alpha \to 0$ (figure 2) such that

$$Z = \begin{cases} 4bs & (s < 0), \quad (26a) \\ -2bs & (s > 0). \quad (26b) \end{cases}$$

Calculations for the velocity components and streamlines are carried out in exactly the same manner as that illustrated in the previous section for each s regime. Details may be found in Adamson (1971).

The constant of integration obtained upon integrating (24) to obtain the streamline or wall shape is in reality a constant of order one plus an arbitrary function of time of the order of the perturbation quantities in the streamline equation. For the present calculation, the arbitrary function of time is included in the initial

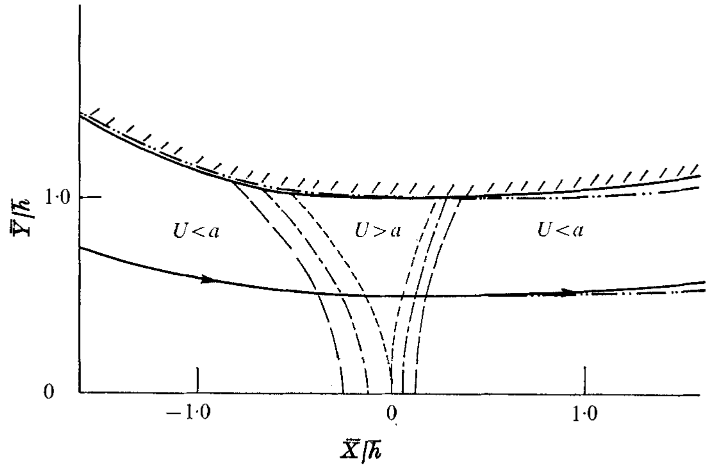

FIGURE 4. Solution for limiting Taylor flow showing variations in the size of supersonic pocket as time increases. $\tau = (k\delta)^{-1}$, $b = k = \frac{1}{2}$, $\gamma = 1{\cdot}4$, $E_1 = 0{\cdot}1$, $\bar{L} = \bar{h}$, $\beta = \frac{1}{4}e^{-2t}$. ———, streamline and wall at $t = \infty$; ··—··, streamline and wall at $t = 0$. Sonic lines: — — —, at $t = 0$; – —–, $t = 0{\cdot}35$; ––––, $t = \infty$.

co-ordinates of the streamline, x_i and y_i. In the examples to be shown, x_i and y_i, for $s \geqslant 0$, have been chosen to be constant and equal to the co-ordinates corresponding to the nozzle throat for the steady-state case. Then, for $s \leqslant 0$, the x_i and y_i values are those which satisfy the equation $s = 0$ and hence are functions of time. Physically, this situation is equivalent to considering a flexible wall pinned at the steady-state throat position. Obviously other conditions, corresponding to different physical problems, could be considered.

Numerical calculations for $b = k = \frac{1}{2}$, $E_1 = 0{\cdot}1$, $\gamma = 1{\cdot}4$, $\bar{L} = \bar{h}$ and $\beta = \frac{1}{4}e^{-2t}$ were carried out and are illustrated in figure 4. This case corresponds physically to that where a flow disturbance is dying out exponentially. The sonic lines ($u = 0$) are again shown at various times. Thus, at $t = 0$, the flow accelerates through sonic velocity, goes through a supersonic pocket and decelerates to subsonic velocities as it proceeds downstream. As time increases, the supersonic pocket decreases in size and finally, as $t \to \infty$, approaches the well-known steady-state limiting Taylor flow shape, where the sonic lines meet at the flow centre-line. The change in wall shape associated with this similarity solution for the decreasing exponential form for β employed in the calculations is shown in figure 4, and is seen to be relatively small compared with the nozzle throat diameter. Finally, it is seen that this limiting Taylor flow corresponds to the case where the fluid acceleration u_x is discontinuous at $s = 0$. However, the curve $s = 0$ can be shown to be a characteristic (Tomotika & Tamada 1950) across which discontinuous derivatives may exist.

5.3. *General Taylor flow*

Taylor flow solutions are those for which $\alpha < 0$ (figure 2). Lines of constant u at a given time are calculated as follows: equation (16b) is solved for z as a function of y for given u and t; solutions of $z = z(s)$ for a given α are employed to give a value of s for each y; finally, with s known (16a) is used to give a value

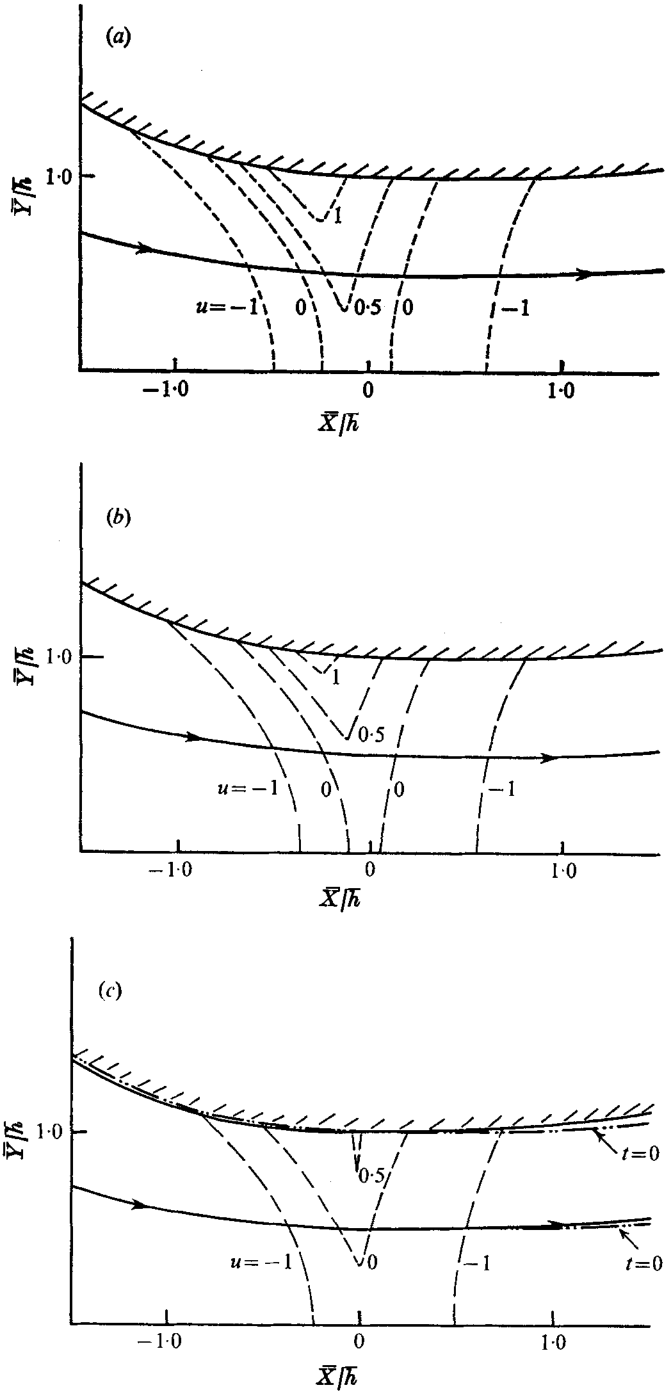

FIGURE 5. Solutions for Taylor flow ($\alpha = -0{\cdot}10$) showing isotachs and streamlines for same parametric values as given in figure 4 and illustrating the manner in which supersonic pocket shrinks as time increases. (*a*) $t = 0$, (*b*) $t = 0{\cdot}35$, (*c*) $t = \infty$. The wall and streamline for $t = 0$ are shown in figure 5(*c*) in order to illustrate the overall variation in wall shape. The sonic line corresponds to $u = 0$, supersonic isotachs to $u > 0$ and the subsonic isotachs to $u < 0$.

of x for each y. An equation for the streamlines can be found by integrating equation (24). Details may be found in Adamson (1971). Again the initial points of the calculations were chosen to be constant such that they correspond with the co-ordinates of the steady-state throat.

Numerical calculations for $\alpha = -0{\cdot}10$ and for the same values of the remaining parameters as those used in the previous example of limiting Taylor flow are shown in figure 5 (*a*) for $t = 0$, in figure 5 (*b*) for $t = 0{\cdot}35$ and in figure 5 (*c*) for $t = \infty$. For this example a more detailed picture of the flow development with time is given in that several constant velocity curves are shown for each time. Thus, subsonic velocities are represented by $u < 0$, sonic by $u = 0$ and supersonic by $u > 0$. In figure 5 (*a*) it is seen that there exists a supersonic pocket which extends across the nozzle. In figure 5 (*b*) it is seen that the size of the supersonic pocket decreases, and that the magnitude of the velocity in the pocket decreases as time goes on. Finally, when the steady-state flow configuration is reached (figure 5 (*c*)) the supersonic pocket exists only along the wall, with subsonic flow in the centre of the nozzle. Again, the flow structure corresponding to that in an asymmetric flow channel is seen by considering the flow between the wall and the streamline pictured, or for that matter, between any two streamlines. The overall streamline variation with time is shown in figure 5 (*c*).

6. Discussion

The solutions presented here exhibit very interesting phenomena when compared with the corresponding steady flows. For example, because the sonic line moves, it is possible for supersonic flow to exist at the point of minimum area. Also, it is possible for a supersonic pocket of flow which initially fills the nozzle to break into two separate pockets, each adjacent to a nozzle wall. If a harmonic function were used for β then these pockets would join and break up once each cycle. This situation would seem to be critical in assessing the possibility of signals being propagated upstream through a nozzle or channel.

Clearly it is desirable to be able to solve the problem of a given channel shape, with or without fluctuating walls, with given initial conditions; this is not possible in general with similarity solutions. However, the similarity solution shown here does cover a few kinds of walls and initial conditions, has the great virtue of computational simplicity and hence appears to be extremely useful in studying and understanding unsteady flow structures in symmetric or asymmetric channels.

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